On the integral representation of g-expectations with terminal constraints

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Abstract. In this paper, we study the integral representation of g-expectations with two kinds of terminal constraints, and obtain the corresponding necessary and sufficient conditions.

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1 Introduction

Pardoux and Peng [15] showed that the following type of nonlinear backward stochastic differential equation (BSDE for short)

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

has a unique solution (Y, Z) under some conditions on g, where ξ is called terminal value and g is called the generator. Based on the solution of BSDEs, Peng [17] introduced the notion of g-expectations $\mathcal{E}_g[\cdot] : L^2(\mathcal{F}_T) \to \mathbb{R}$, which is the first kind of dynamically consistent nonlinear expectations. Moreover, Coquet et al. [7] proved that any dynamically consistent nonlinear expectation on $L^2(\mathcal{F}_T)$ under certain conditions is g-expectation.

One problem of g-expectation is to find the condition of g under which the following integral representation

$$\mathcal{E}_g[\xi] = \int_{-\infty}^0 (\mathcal{E}_g[I_{\{\xi \geq t\}}] - 1)dt + \int_0^\infty \mathcal{E}_g[I_{\{\xi \geq t\}}]dt \tag{1}$$

holds. Chen et al. [3] proved that the integral representation (1) holds for each $\xi \in L^2(\mathcal{F}_T)$ if and only if $\mathcal{E}_g[\cdot]$ is a classical linear expectation under the assumptions: g is continuous in t and W is 1-dimensional Brownian motion. Without these assumptions on g and W, Hu [12, 13] showed that the above

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result on integral representation (1) for each $\xi \in L^2(\mathcal{F}_T)$ still holds. For the integral representation (1) with terminal constraints on $\xi = \Phi(X_T)$, where Φ is a monotonic function and X is a solution of stochastic differential equation (SDE for short), Chen et al. [5, 4] obtained a necessary and sufficient condition under the above assumptions on g and W, and gave a sufficient condition for multi-dimensional Brownian motion.

In this paper, we want to study the integral representation (1) with the following two kinds of terminal constraints on $\xi = \Phi(X_T)$: one is for the monotonic Φ, the other is for the measurable Φ. Specially, we make further research to the structure of Z in the BSDE and apply it to obtain the corresponding necessary and sufficient conditions without the above assumptions on g and W, which is weaker than the sufficient condition in [4] (see Remark 9 in Section 3 for detailed explanation). Furthermore, this method can be extended to solve more general terminal constraints on ξ.

This paper is organized as follows: In Section 2, we recall some basic results of BSDEs and g-expectations. The main result is stated and proved in Section 3.

2 Preliminaries

Let $(W_t)_{t\geq 0} = (W_t^1, \dots, W_t^d)_{t\geq 0}$ be a d-dimensional standard Brownian motion defined on a completed probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{0\leq t\leq T}$ be the natural filtration generated by this Brownian motion, i.e.,

$$\mathcal{F}_t := \sigma\{W_s : s \leq t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets. Fix $T > 0$, we denote by $L^2(\mathcal{F}_t; \mathbb{R}^m)$, $t \in [0, T]$, the set of all $\mathbb{R}^m$-valued square integrable $\mathcal{F}_t$-measurable random vectors and $L^2(0, T; \mathbb{R}^m)$ the space of all progressively measurable, $\mathbb{R}^m$-valued processes $(a_t)_{t\in[0,T]}$ with $E[\int_0^T |a_t|^2 dt] < \infty$.

We consider the following forward-backward stochastic differential equations:

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s, \ s \in [t, T], \\ X_t^{t,x} = x \in \mathbb{R}^n, \end{cases} \tag{2}$$

$$y_s^{t,x} = \Phi(X_T^{t,x}) + \int_s^T g(r, y_r^{t,x}, z_r^{t,x})dr - \int_s^T z_r^{t,x} dW_r. \tag{3}$$

In this paper, we use the following assumptions:

(S1) $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are measurable.

(S2) There exists a constant $K_1 \geq 0$ such that

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq K_1|x - x'|, \ \forall t \leq T, x, x' \in \mathbb{R}^n.$$

(S3) $\int_0^T (|b(t, 0)|^2 + |\sigma(t, 0)|^2)dt < \infty$.

(H1) $g : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is measurable.

(H2) There exists a constant $K_2 \geq 0$ such that

$$|g(t,y,z) - g(t,y',z')| \leq K_2(|y-y'| + |z-z'|), \ \forall t \leq T, y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d.$$

(H3) $g(t,y,0) \equiv 0$ for each $(t,y) \in [0,T] \times \mathbb{R}$.

(H3') $\int_0^T |g(t,0,0)|^2 dt < \infty$.

(H4) $\Phi : \mathbb{R}^n \to \mathbb{R}$ is measurable and satisfies $\Phi(X_T^{t,x}) \in L^2(\mathcal{F}_T)$.

Remark 1 *Obviously, (H3) implies (H3').*

It is well-known that the SDE (2) has a unique solution $(X_s^{t,x})_{s\in[t,T]} \in L^2(t,T;\mathbb{R}^n)$ under the assumptions (S1)-(S3). Under the assumptions (H1), (H2), (H3') and (H4), Pardoux and Peng [15] showed that the BSDE (3) has a unique solution $(y_s^{t,x}, z_s^{t,x})_{s\in[0,T]} \in L^2(0,T;\mathbb{R}^{1+d})$. Moreover, the following result holds.

Theorem 2 *([10, 16]) Suppose (S1)-(S3), (H1), (H2), (H3') and (H4) hold. If b, σ, g and $\Phi \in C_b^{1,3}$, then*

(i) *$u(t,x) := y_t^{t,x} \in C^{1,2}([0,T] \times \mathbb{R}^n)$ and solves the following PDE:*

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + g(t,u(t,x),\sigma^T(t,x)\partial_x u(t,x)) = 0, \\ u(T,x) = \Phi(x), \end{cases}$$

where

$$\mathcal{L}u(t,x) = \frac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^T)_{ij}(t,x)\partial^2_{x_i x_j} u(t,x) + \sum_{i=1}^n b_i(t,x)\partial_{x_i} u(t,x).$$

(ii) *$z_s^{t,x} = \sigma^T(s, X_s^{t,x})\partial_x u(s, X_s^{t,x})$, $s \in [t,T]$.*

Remark 3 *For notation simplicity, when $t = 0$ and only one x, we write $(X_t, y_t, z_t)_{t\in[0,T]}$ for the solution of SDE (2) and BSDE (3) in the following.*

Using the solution of BSDE, Peng [17] proposed the following consistent nonlinear expectations.

Definition 4 *Suppose g satisfies (H1)-(H3). Let $(y_t, z_t)_{t\in[0,T]}$ be the solution of BSDE (3) with terminal value $\xi \in L^2(\mathcal{F}_T)$, i.e.,*

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dW_s.$$

Define

$$\mathcal{E}_g[\xi|\mathcal{F}_t] := y_t \quad \text{for each } t \in [0,T].$$

$\mathcal{E}_g[\xi|\mathcal{F}_t]$ is called the conditional g-expectation of ξ with respect to $\mathcal{F}_t$. In particular, if $t = 0$, we write $\mathcal{E}_g[\xi]$ which is called the g-expectation of ξ.

Remark 5 *The assumption (H3) is important in the definition of g-expectation. In particular, under the assumptions (H1)-(H3), if $\xi \in L^2(\mathcal{F}_{t_0})$ with $t_0 < T$, then $\mathcal{E}_g[\xi|\mathcal{F}_t] = \xi$ for $t \in [t_0, T]$.*

The following standard estimates of BSDEs can be found in [10, **?**, 1].

Proposition 6 *Suppose g_1 and g_2 satisfy (H1), (H2) and (H3'). Let $(y^i_t, z^i_t)_{t\in[0,T]}$ be the solution of BSDE (3) with the generator g_i and terminal value $\xi_i \in L^2(\mathcal{F}_T)$, $i = 1, 2$. Then there exists a constant $C > 0$ depending on K_2 and T such that*

$$E[\sup_{0\le t\le T} |y^1_t - y^2_t|^2 + \int_0^T |z^1_t - z^2_t|^2 dt] \le CE[|\xi^1 - \xi^2|^2 + \int_0^T |\bar{g}_t|^2 dt],$$

where $\bar{g}_t = g_1(t, y^1_t, z^1_t) - g_2(t, y^1_t, z^1_t)$.

Assume g satisfies (H1)-(H3), set

$$V_g(A) := \mathcal{E}_g[I_A] \quad \text{for each } A \in \mathcal{F}_T.$$

It is easy to verify that $V_g(\cdot)$ is a capacity, i.e., (i) $V_g(\emptyset) = 0$, $V_g(\Omega) = 1$; (ii) $V_g(A) \le V_g(B)$ for each $A \subset B$. The corresponding Choquet integral (see [6]) is defined as follows:

$$\mathcal{C}_g[\xi] := \int_{-\infty}^0 [V_g(\xi \ge t) - 1]dt + \int_0^\infty V_g(\xi \ge t)dt \quad \text{for each } \xi \in L^2(\mathcal{F}_T).$$

It is easy to check that $\mathcal{C}_g[I_A] = \mathcal{E}_g[I_A]$ for each $A \in \mathcal{F}_T$. Moreover, $|\mathcal{C}_g[\xi]| < \infty$ for each $\xi \in L^2(\mathcal{F}_T)$ (see [11]).

Definition 7 *Two random variables ξ and η are called comonotonic if*

$$[\xi(\omega) - \xi(\omega')][\eta(\omega) - \eta(\omega')] \ge 0 \quad \textit{for each } \omega, \omega' \in \Omega.$$

The following properties of Choquet integral can be found in [6, 8, 9].

(1) Monotonicity: If $\xi \ge \eta$, then $\mathcal{C}_g[\xi] \ge \mathcal{C}_g[\eta]$.

(2) Positive homogeneity: If $\lambda \ge 0$, then $\mathcal{C}_g[\lambda\xi] = \lambda\mathcal{C}_g[\xi]$.

(3) Translation invariance: If $c \in \mathbb{R}$, then $\mathcal{C}_g[\xi + c] = \mathcal{C}_g[\xi] + c$.

(4) Comonotonic additivity: If ξ and η are comonotonic, then $\mathcal{C}_g[\xi + \eta] = \mathcal{C}_g[\xi] + \mathcal{C}_g[\eta]$.

3 Main result

Suppose $n = 1$, we define

$$\mathcal{H} := \{\xi : \exists b, \sigma \text{ satisfying (S1)-(S3) and } x \text{ such that } \xi = X_T^{0,x}\}.$$
$$\mathcal{H}_1 := \{\Phi(\xi) \in L^2(\mathcal{F}_T) : \Phi \text{ is monotonic and } \xi \in \mathcal{H}\}.$$
$$\mathcal{H}_2 := \{\Phi(\xi) \in L^2(\mathcal{F}_T) : \Phi \text{ is measurable and } \xi \in \mathcal{H}\}.$$

The elements in $\mathcal{H}_1$ and $\mathcal{H}_2$ can be seen as the contingent claims of European option. Now we give our main theorem.

Theorem 8 *Suppose g satisfies (H1)-(H3). Then*

(i) *$\mathcal{E}_g[\cdot] = \mathcal{C}_g[\cdot]$ on $\mathcal{H}_1$ if and only if g is independent of y and is positively homogeneous in z, i.e., $g(t, \lambda z) = \lambda g(t, z)$ for all $\lambda \geq 0$;*

(ii) *$\mathcal{E}_g[\cdot] = \mathcal{C}_g[\cdot]$ on $\mathcal{H}_2$ if and only if g is independent of y and is homogeneous in z, i.e., $g(t, \lambda z) = \lambda g(t, z)$ for all $\lambda \in \mathbb{R}$.*

Remark 9 *In [4], Chen et al. showed that $\mathcal{E}_g[\cdot] = \mathcal{C}_g[\cdot]$ on $\mathcal{H}_1$ under the assumption that g is positively additive, i.e., $g(t, z_1+z_1', \ldots, z_d+z_d') = g(t, z_1, \ldots, z_d)+g(t, z_1', \ldots, z_d')$ for $z_i z_i' \geq 0$, $i = 1, \ldots, d$. Obviously, this condition on g is stronger than positive homogeneity. For example, $g(z) = |z|$ is not positively additive, but is positively homogeneous.*

In order to prove this theorem, we need the following lemmas.

Lemma 10 *Suppose g satisfies (H1)-(H3). Then for each given $p \in (1, 2)$, there exists a constant $L > 0$ depending on p, K_2 and T such that for each ξ, $\eta \in L^2(\mathcal{F}_T)$,*

$$|\mathcal{C}_g[\xi] - \mathcal{C}_g[\eta]| \leq L(1 + (E[|\xi|^2 + |\eta|^2])^{\frac{1}{2p}})(E[|\xi - \eta|^2])^{\frac{1}{2p}}.$$

In particular, for each $\xi \in L^2(\mathcal{F}_T)$, we have $\mathcal{C}_g[(\xi \wedge N) \vee (-N)] \to \mathcal{C}_g[\xi]$ as $N \to \infty$.

Proof. For each given $p \in (1, 2)$, by Proposition 3.2 in Briand et al. [2], there exists a constant $L_1 > 0$ depending on p, K_2 and T such that for each ξ, $\eta \in L^2(\mathcal{F}_T)$,

$$|\mathcal{E}_g[\xi] - \mathcal{E}_g[\eta]| \leq L_1(E[|\xi - \eta|^p])^{\frac{1}{p}}.$$

Set $\bar{g}(t, y, z) = -g(t, 1 - y, -z)$, it is easy to check that $1 - V_g(A) = V_{\bar{g}}(A^c)$. Thus $\mathcal{C}_g[\xi] = \mathcal{C}_g[\xi^+] - \mathcal{C}_{\bar{g}}[\xi^-]$. From this we only need to prove the result for $\xi \geq 0$ and $\eta \geq 0$. We have

$$\begin{aligned} |\mathcal{C}_g[\xi] - \mathcal{C}_g[\eta]| &\leq \int_0^\infty |\mathcal{E}_g[I_{\{\xi \geq t\}}] - \mathcal{E}_g[I_{\{\eta \geq t\}}]| dt \\ &\leq L_1 \int_0^\infty (E[|I_{\{\xi \geq t\}} - I_{\{\eta \geq t\}}|^p])^{\frac{1}{p}} dt \\ &= L_1 \int_0^\infty (E[|I_{\{\xi \geq t\}} - I_{\{\eta \geq t\}}|])^{\frac{1}{p}} dt, \end{aligned}$$

$$\int_0^1 (E[|I_{\{\xi\geq t\}} - I_{\{\eta\geq t\}}|])^{\frac{1}{p}} dt \leq (E[\int_0^1 |I_{\{\xi\geq t\}} - I_{\{\eta\geq t\}}| dt])^{\frac{1}{p}}$$
$$= (E[\int_0^1 I_{\{\xi\wedge\eta < t\leq \xi\vee\eta\}} dt])^{\frac{1}{p}}$$
$$\leq (E[|\xi - \eta|])^{\frac{1}{p}}$$
$$\leq (E[|\xi - \eta|^2])^{\frac{1}{2p}},$$

$$\int_1^\infty (E[|I_{\{\xi\geq t\}} - I_{\{\eta\geq t\}}|])^{\frac{1}{p}} dt \leq (\int_1^\infty t^{-\frac{q}{p}} dt)^{\frac{1}{q}} (E[\int_1^\infty t|I_{\{\xi\geq t\}} - I_{\{\eta\geq t\}}| dt])^{\frac{1}{p}}$$
$$= (\frac{p-1}{2-p})^{\frac{p-1}{p}} (E[\int_1^\infty t I_{\{\xi\wedge\eta < t\leq \xi\vee\eta\}} dt])^{\frac{1}{p}}$$
$$\leq (\frac{p-1}{2-p})^{\frac{p-1}{p}} (\frac{1}{2} E[|\xi^2 - \eta^2|])^{\frac{1}{p}}$$
$$\leq (\frac{p-1}{2-p})^{\frac{p-1}{p}} (\frac{1}{2})^{\frac{1}{2p}} (E[|\xi|^2 + |\eta|^2])^{\frac{1}{2p}} (E[|\xi - \eta|^2])^{\frac{1}{2p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Thus we obtain the result. $\square$

Lemma 11 *Let b, σ satisfy (S1)-(S3), g satisfy (H1)-(H3) and $\Phi \in C_b^3$. Then there exist b_k, σ_k, $g_k \in C_b^{1,3}$, $k \geq 1$, such that*

$$E[\sup_{t\in[0,T]} |X_t^k - X_t|^2 + \int_0^T (|\sigma_k(t, X_t^k) - \sigma(t, X_t)|^2 + |z_t^k - z_t|^2) dt] \to 0,$$

where $(X_t, y_t, z_t)_{t\in[0,T]}$ is the solution corresponding to b, σ, g and $(X_t^k, y_t^k, z_t^k)_{t\in[0,T]}$ is the solution corresponding to b_k, σ_k, g_k.

Proof. By the standard estimates of SDEs and Proposition 6, we only need to prove the result for bounded b, σ and g. For any function $h(u)$, $u \in \mathbb{R}^m$, we will denote, for each $\varepsilon > 0$,

$$h_\varepsilon(u) = \int_{\mathbb{R}^m} h(u - v)\varepsilon^{-m}\varphi(\frac{v}{\varepsilon}) dv,$$

where φ is the mollifier in $\mathbb{R}^m$ defined by $\varphi(u) = \exp(-\frac{1}{1-|u|^2}) I_{\{|u|<1\}}$. By this definition, it is easy to check that b_ε, σ_ε and g_ε satisfy (S2) and (H2) with the same Lipschitz constant. Also, we have b_ε, σ_ε, $g_\varepsilon \in C_b^{1,3}$ and $(b_\varepsilon, \sigma_\varepsilon, g_\varepsilon) \to (b, \sigma, g)$ a.e. in t for each fixed $(x, y, z) \in \mathbb{R}^{2+d}$. Thus by the diagonal method, we can choose a sequence b_k, σ_k, $g_k \in C_b^{1,3}$ such that $(b_k, \sigma_k, g_k) \to (b, \sigma, g)$ for every $(x, y, z) \in \mathbb{Q}^{2+d}$ a.e. in t. By the Lipschitz condition, we get $(b_k, \sigma_k, g_k) \to (b, \sigma, g)$ for every $(x, y, z) \in \mathbb{R}^{2+d}$ a.e. in t. By the estimates of SDEs, we obtain

$$E[\sup_{t\in[0,T]} |X_t^k - X_t|^2] \leq L_2 E[\int_0^T (|b_k(t, X_t) - b(t, X_t)|^2 + |\sigma_k(t, X_t) - \sigma(t, X_t)|^2) dt],$$

where the constant L_2 depending on K_1 and T. By the bounded dominated convergence theorem, we can get $E[\sup_{t\in[0,T]}|X_t^k - X_t|^2] \to 0$. From this, it is easy to deduce that $E[\int_0^T |\sigma_k(t,X_t^k) - \sigma(t,X_t)|^2 dt] \to 0$. By Proposition 6, we can easily obtain $E[\int_0^T |z_t^k - z_t|^2 dt] \to 0$. $\square$

We now prove the main theorem.

Proof of Theorem 8. We first prove that the condition on g is necessary, and then it is sufficient.

(i) Necessity. We first prove the result for the case $d = 1$. For this we choose $b(s,x) = 0$, $\sigma(s,x) = zI_{[t,t+\varepsilon]}(s)$ and $\Phi(x) = x$, where $z \in \mathbb{R}$, $t < T$ and $\varepsilon > 0$ are given. Then

$$\mathcal{H}_1 \supset \{y + z(W_{t+\varepsilon} - W_t) : \forall y, z \in \mathbb{R}, t < T, \varepsilon > 0\}.$$

Since $\mathcal{E}_g[\cdot] = \mathcal{C}_g[\cdot]$ on $\mathcal{H}_1$ and g is deterministic, by the properties of $\mathcal{C}_g[\cdot]$ we can get

$$\mathcal{E}_g[y + z(W_{t+\varepsilon} - W_t)|\mathcal{F}_t] = \mathcal{E}_g[y + z(W_{t+\varepsilon} - W_t)] = \mathcal{E}_g[z(W_{t+\varepsilon} - W_t)|\mathcal{F}_t] + y,$$

$$\mathcal{E}_g[\lambda z(W_{t+\varepsilon} - W_t)|\mathcal{F}_t] = \lambda\mathcal{E}_g[z(W_{t+\varepsilon} - W_t)|\mathcal{F}_t] \text{ for } \lambda \geq 0.$$

By Lemma 2.1 in Jiang [14], we can obtain that g is independent of y and $g(t,\lambda z) = \lambda g(t,z)$ for all $\lambda \geq 0$. For the case $d > 1$. For each given $a \in \mathbb{R}^d$ with $|a| = 1$, we define W^a by $W_t^a = a \cdot W_t$ and $g^a : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $g^a(t,y,z) = g(t,y,az)$. It is easy to check that $\mathcal{E}_g[\xi] = \mathcal{E}_{g^a}[\xi]$ and $\mathcal{C}_g[\xi] = \mathcal{C}_{g^a}[\xi]$ for $\xi \in L^2(\mathcal{F}_T^a)$, where $\mathcal{F}_T^a := \sigma\{W_t^a : t \leq T\} \vee \mathcal{N}$. Thus by applying the method of $d = 1$, we can obtain g^a is independent of y and is positively homogeneous in z for each given $a \in \mathbb{R}^d$ with $|a| = 1$, which implies the necessary condition on g.

Sufficiency. By Proposition 6 and Lemma 10, we only need to prove the result for bounded and monotonic Φ. The proof is divided into two steps.

Step 1. Let $(X_t)_{t\in[0,T]}$ be the solution of SDE (2) corresponding to b and σ satisfying (S1)-(S3) and let $\phi_i \in C_b^3(\mathbb{R})$, $i = 1, \ldots, N$, be non decreasing functions. We assert that

$$\mathcal{E}_g[\sum_{i=1}^N \phi_i(X_T)] = \sum_{i=1}^N \mathcal{E}_g[\phi_i(X_T)]. \tag{4}$$

Let $(y_t^i, z_t^i)_{t\in[0,T]}$, $i = 1, \ldots, N$, be the solution of the following BSDEs:

$$y_t^i = \phi_i(X_T) + \int_t^T g(s, z_s^i)ds - \int_t^T z_s^i dW_s. \tag{5}$$

By Lemma 11, we can choose b_k, σ_k, $g_k \in C_b^{1,3}$, $k \geq 1$, such that

$$E[\int_0^T (|\sigma_k(t,X_t^k) - \sigma(t,X_t)|^2 + |z_t^{i,k} - z_t^i|^2)dt] \to 0,\ i = 1, \ldots, N,$$

where $(X_t^k, y_t^{i,k}, z_t^{i,k})_{t\in[0,T]}$ is the solution corresponding b_k, σ_k, g_k and terminal value $\phi_i(X_T^k)$. From this we can get

$$z_t^{i,k} \to z_t^i,\ \ \sigma_k(t, X_t^k) \to \sigma(t, X_t) \ \ dP \times dt\text{-a.s.}. \tag{6}$$

On the other hand, it follows from Theorem 2 that

$$z_t^{i,k} = \sigma_k^T(t, X_t^k)\partial_x u^{i,k}(t, X_t^k), \tag{7}$$

where $u^{i,k}(t,x) := y_t^{i,k;t,x}$. By comparison theorem of SDE and BSDE, it is easy to verify that $u^{i,k}(t,x)$ is non decreasing in x, which implies $\partial_x u^{i,k}(t, X_t^k) \geq 0$. Thus by combining equation (6) and (7), we obtain that there exist progressive processes $D_t^i \geq 0$, $i = 1, \ldots, N$, such that

$$z_t^i = \sigma^T(t, X_t)D_t^i.$$

Note that g is positively homogeneous in z, then we get

$$\begin{aligned}\sum_{i=1}^N g(t, z_t^i) &= \sum_{i=1}^N g(t, \sigma^T(t, X_t)D_t^i) = g(t, \sigma^T(t, X_t))\sum_{i=1}^N D_t^i \\ &= g(t, \sigma^T(t, X_t)\sum_{i=1}^N D_t^i) = g(t, \sum_{i=1}^N z_t^i).\end{aligned} \tag{8}$$

Set

$$Y_t = \sum_{i=1}^N y_t^i,\ \ Z_t = \sum_{i=1}^N z_t^i,$$

then by combining equation (5) and (8), we can get

$$Y_t = \sum_{i=1}^N \phi_i(X_T) + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dW_s.$$

By the definition of g-expectation, we obtain equation (4).

Step 2. Let $(X_t)_{t\in[0,T]}$ be as in Step 1 and let Φ be a bounded and monotonic function. Note that for each $\xi \in L^2(\mathcal{F}_T)$ and $c \in \mathbb{R}$,

$$\mathcal{E}_g[\xi + c] = \mathcal{E}_g[\xi] + c,\ \mathcal{C}_g[\xi + c] = \mathcal{C}_g[\xi] + c,$$

then we only need to prove the result for $\Phi \geq 0$. Since the analysis of non increasing Φ is the same as in non decreasing Φ, we only prove the case for non decreasing Φ with $0 \leq \Phi < M$, where $M > 0$ is a constant. For each given $N > 0$, we set

$$\Phi_N(x) = \sum_{i=1}^N \frac{(i-1)M}{N} I_{\{\frac{(i-1)M}{N} \leq \Phi < \frac{iM}{N}\}} = \sum_{i=1}^N \frac{M}{N} I_{\{\Phi \geq \frac{iM}{N}\}}.$$

It is easy to check that $E[|\Phi_N(X_T)-\Phi(X_T)|^2]\leq(\frac{M}{N})^2\to 0$ as $N\to\infty$. Thus by Proposition 6 and Lemma 10, we get

$$\mathcal{E}_g[\Phi_N(X_T)]\to\mathcal{E}_g[\Phi(X_T)],\ \mathcal{C}_g[\Phi_N(X_T)]\to\mathcal{C}_g[\Phi(X_T)] \text{ as } N\to\infty. \tag{9}$$

For each fixed $N>0$, noting that Φ is non decreasing, then $\{\Phi\geq\frac{iM}{N}\}$ is $[a_i,\infty)$ or (a_i,∞), where a_i is a constant. For each $\varepsilon>0$, we define

$$\psi^1_{i,\varepsilon}(x)=\int_{\mathbb{R}}I_{[a_i-\varepsilon,\infty)}(x-v)\frac{1}{\varepsilon}\varphi(\frac{v}{\varepsilon})dv,\psi^2_{i,\varepsilon}(x)=\int_{\mathbb{R}}I_{(a_i+\varepsilon,\infty)}(x-v)\frac{1}{\varepsilon}\varphi(\frac{v}{\varepsilon})dv,$$

where $\varphi(v)=\exp(-\frac{1}{1-|v|^2})I_{\{|v|<1\}}$. It is easy to check that $\psi^1_{i,\varepsilon}$, $\psi^2_{i,\varepsilon}\in C^3_b(\mathbb{R})$ are non decreasing and satisfy $\psi^1_{i,\varepsilon}\downarrow I_{[a_i,\infty)}$, $\psi^2_{i,\varepsilon}\uparrow I_{(a_i,\infty)}$ as $\varepsilon\downarrow 0$. Thus we can choose non decreasing $\phi^k_i\in C^3_b(\mathbb{R})$, $k\geq 1$, such that $E[|\phi^k_i(X_T)-I_{\{\Phi\geq\frac{iM}{N}\}}(X_T)|^2]\to 0$ as $k\to\infty$, which implies

$$E[|\Phi_N(X_T)-\frac{M}{N}\sum_{i=1}^N\phi^k_i(X_T)|^2]\to 0 \text{ as } k\to\infty.$$

By Step 1, Proposition 6 and properties of Choquet integral, we can obtain

$$\begin{aligned}\mathcal{E}_g[\Phi_N(X_T)]&=\lim_{k\to\infty}\mathcal{E}_g[\frac{M}{N}\sum_{i=1}^N\phi^k_i(X_T)]=\lim_{k\to\infty}\frac{M}{N}\mathcal{E}_g[\sum_{i=1}^N\phi^k_i(X_T)]\\&=\frac{M}{N}\sum_{i=1}^N\lim_{k\to\infty}\mathcal{E}_g[\phi^k_i(X_T)]=\frac{M}{N}\sum_{i=1}^N\mathcal{E}_g[I_{\{\Phi\geq\frac{iM}{N}\}}(X_T)]\\&=\frac{M}{N}\sum_{i=1}^N\mathcal{C}_g[I_{\{\Phi\geq\frac{iM}{N}\}}(X_T)]=\mathcal{C}_g[\Phi_N(X_T)].\end{aligned}$$

Thus by (9), we get $\mathcal{E}_g[\Phi(X_T)]=\mathcal{C}_g[\Phi(X_T)]$. The proof of (i) is complete.

(ii) Necessity. For the case $d=1$, since $\mathcal{H}_2\supset\mathcal{H}_1$, we can get that g is independent of y and is positively homogeneous in z by (i). On the other hand,

$$\{l_1I_{\{W_T-W_t\geq a\}}+l_2I_{\{b\geq W_T-W_t\geq a\}}:t<T,a<b,a,b,l_1,l_2\in\mathbb{R}\}\subset\mathcal{H}_2,$$

by the proof of Lemma 9 in [12], we can obtain $g(t,z)=g(t,1)z$. For the case $d>1$, the proof is the same as (i).

Sufficiency. By the similar analysis as in (i), for each $\phi_i\in C^3_b(\mathbb{R})$, $i=1,\ldots,N$, we can get

$$\mathcal{E}_g[\sum_{i=1}^N\phi_i(X_T)]=\sum_{i=1}^N\mathcal{E}_g[\phi_i(X_T)].$$

The same analysis as in (i), we only need to prove the result for

$$\Phi(x)=\sum_{i=1}^N b_iI_{A_i}(x),$$

where $b_i \geq 0$, $A_i \in \mathcal{B}(\mathbb{R})$ and $A_i \supset A_{i+1}$. Set

$$P_{X_T}(A) := P(X_T^{-1}(A)) \quad \text{for } A \in \mathcal{B}(\mathbb{R}),$$

then by Lusin's theorem, we can choose $\phi_i^k \in C_b^3(\mathbb{R})$, $k \geq 1$, such that

$$E[|\phi_i^k(X_T) - I_{A_i}(X_T)|^2] = E_{P_{X_T}}[|\phi_i^k(x) - I_{A_i}(x)|^2] \to 0 \quad \text{as } k \to \infty.$$

Thus we obtain $\mathcal{E}_g[\Phi(X_T)] = \mathcal{C}_g[\Phi(X_T)]$ as in (i). The proof is complete. $\square$

In the following, we consider the case $n > 1$. We give the following assumptions on σ in SDE (2).

(S4) There exists a $k \leq d$ such that $\sigma_i(t,x) = (\tilde{\sigma}(t,x), 0, \ldots, 0)$ for $i = 1, \ldots, n$, where σ_i is the i-th row of σ and $\tilde{\sigma} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{1\times k}$.

(S5) There exists a $k \leq d$ such that $\sigma_i(t,x) = (\tilde{\sigma}(t,x), \tilde{\sigma}_i(t,x))$ for $i = 1, \ldots, n$, where σ_i is the i-th row of σ, $\tilde{\sigma} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{1\times k}$ and $\tilde{\sigma}_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{1\times(d-k)}$.

Set

$\mathcal{H}_3 := \{\xi : \exists b, \sigma$ satisfying (S1)-(S3), (S4) and $x \in \mathbb{R}^n$ such that $\xi = X_T^{0,x}\}$.
$\mathcal{H}_4 := \{\xi : \exists b, \sigma$ satisfying (S1)-(S3), (S5) and $x \in \mathbb{R}^n$ such that $\xi = X_T^{0,x}\}$.
$\mathcal{H}_5 := \{\Phi(\xi) \in L^2(\mathcal{F}_T) : \Phi$ is measurable on $\mathbb{R}^n$ and $\xi \in \mathcal{H}_3\}$.
$\mathcal{H}_6 := \{\Phi(\xi) \in L^2(\mathcal{F}_T) : \Phi$ is measurable on $\mathbb{R}^n$ and $\xi \in \mathcal{H}_4\}$.

By the same analysis as in the proof of Theorem 8 and the method in the proof of main result in [12, 13], we can obtain the following corollary.

Corollary 12 *Suppose g satisfies (H1)-(H3). Then*

(i) $\mathcal{E}_g[\cdot] = \mathcal{C}_g[\cdot]$ *on $\mathcal{H}_5$ if and only if $\tilde{g}$ is independent of y and is homogeneous in $\tilde{z}$, where $\tilde{g}(t,y,\tilde{z}) := g(t,y,(\tilde{z},0,\ldots,0))$ for $(t,y,\tilde{z}) \in [0,T] \times \mathbb{R}^{1+k}$;*

(ii) $\mathcal{E}_g[\cdot] = \mathcal{C}_g[\cdot]$ *on $\mathcal{H}_6$ if and only if g is independent of y, $g(t,(\tilde{z},z')) = g_1(t,\tilde{z}) + g_2(t,z')$ for $\tilde{z} \in \mathbb{R}^k$, $z' \in \mathbb{R}^{d-k}$, g_1 is homogeneous in $\tilde{z}$ and g_2 is linear in z'.*

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